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## ON THE HARDY–CARLEMAN INEQUALITY FOR A NEGATIVE EXPONENT

ZHANG DAOXIANG AND PING YAN

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*Abstract.* In this paper we settle an open problem raised by B. Yang (2005, *Taiwanese Journal of Mathematics* 9, 469–475), by using Hölder’s and Bernoulli’s inequalities. We give a strengthened Hardy–Carleman inequality for a negative exponent.

### 1. Introduction

The following inequality of Hardy’s is well known [2, Chap. 9.12]:

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^n a_k^{\frac{1}{p}} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n. \quad (1)$$

Here  $p > 1$ ,  $a_n \geq 0$  ( $n \in \mathbf{N}$ ) and  $0 < \sum_{n=1}^{\infty} a_n < \infty$ .

The constant  $(\frac{p}{p-1})^p$  in (1) is the best possible. As  $p$  tends to infinity the inequality (1) reduces to Carleman’s inequality

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n, \quad (2)$$

where the constant  $e$  in (2) is still the best possible [2, Chap. 9.12]. The inequalities (1) and (2) are important in analysis and its applications [3].

In [8], we proved the following strengthened version of (2).

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left( 1 + \frac{1}{n + \frac{1}{5}} \right)^{-\frac{1}{2}} a_n. \quad (3)$$

Some other strengthened versions of (2) and related results can be found in [1, 7, 8, 9, 11].

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If we set  $p = \frac{1}{r}$  in (1), then we have  $0 < r < 1$ , and (1) is equivalent to the inequality

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^n a_k^r \right)^{1/r} < \left( \frac{1}{1-r} \right)^{1/r} \sum_{n=1}^{\infty} a_n, \quad (4)$$

where the constant  $(\frac{1}{1-r})^{1/r}$  is the best possible.

Thanh et al. [6] discussed (4) for  $r \in (-\infty, 0)$ , and proved the following result: If  $a_n \geq 0$  for  $n \in \mathbf{N}$  and  $0 < \sum_{n=1}^{\infty} a_n < \infty$ , then

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^n a_k^r \right)^{1/r} < \left( \frac{1}{1-r} \right)^{1/r} \sum_{n=1}^{\infty} a_n \quad (5)$$

if  $-1 \leq r < 0$  and

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^n a_k^r \right)^{1/r} < \frac{r}{r-1} 2^{\frac{r-1}{r}} \sum_{n=1}^{\infty} a_n \quad (6)$$

if  $r < -1$ .

If we replace  $r$  by  $-r$ , in (5) and (6) we obtain

$$\sum_{n=1}^{\infty} \left( \frac{n}{\sum_{k=1}^n a_k^{-r}} \right)^{1/r} < (1+r)^{1/r} \sum_{n=1}^{\infty} a_n \quad (7)$$

if  $0 < r \leq 1$  and

$$\sum_{n=1}^{\infty} \left( \frac{n}{\sum_{k=1}^n a_k^{-r}} \right)^{1/r} < \frac{r}{1+r} 2^{\frac{1+r}{r}} \sum_{n=1}^{\infty} a_n \quad (8)$$

if  $1 < r < \infty$ .

Recently, Yang [10] proved that the constant  $(1+r)^{1/r}$  in (7) is the best possible for  $0 < r \leq 1$ . At the end of paper [10], Yang posed the question:

Is the constant factor  $\frac{r}{1+r} 2^{\frac{1+r}{r}}$  in (8) the best possible or not for  $1 < r < \infty$ ?

In this paper we solve this problem. We give a strengthened Hardy-Carleman inequality for a negative exponent.

## 2. Main results

In this section, we prove the following theorem.

**THEOREM 2.1.** *Let  $1 < r < \infty$ ,  $a_n \geq 0$  ( $n \in \mathbf{N}$ ) and  $0 < \sum_{n=1}^{\infty} a_n < \infty$ . Then*

$$\sum_{n=1}^{\infty} \left( \frac{n}{\sum_{k=1}^n a_k^{-r}} \right)^{1/r} < \frac{1}{r} (1+r)^{\frac{1+r}{r}} \sum_{n=1}^{\infty} a_n. \quad (9)$$

To prove Theorem 2.1, we use Hölder's inequality (with negative exponent  $p$ ) (see [4, page 29]) and Bernoulli's inequality. For the convenience of the reader we start by recalling these results.

LEMMA 2.1. (Hölder's inequality) *Suppose that  $p < 0$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $f(x), g(x) \geq 0$  for  $x \in [a, b]$ , and  $f \in L^p[a, b]$ ,  $g \in L^q[a, b]$ . Then*

$$\int_a^b f(t)g(t)dt \geq \left( \int_a^b f^p(t)dt \right)^{1/p} \left( \int_a^b g^q(t)dt \right)^{1/q},$$

where equality holds only if there exist real numbers  $\alpha$  and  $\beta$ , such that  $\alpha^2 + \beta^2 > 0$ , and  $\alpha f^p(x) = \beta g^q(x)$ , a.e. in  $[a, b]$ .

LEMMA 2.2. (Bernoulli inequality) *Suppose that  $x \geq -1$  and  $0 < \alpha < 1$ . Then*

$$(1+x)^\alpha \leq 1 + \alpha x,$$

where equality holds if and only if  $x = 0$ .

We also need the following lemmas.

LEMMA 2.3. *Suppose that  $0 < \alpha < 1$  and  $x > 0$ . Then*

$$1 + \frac{\alpha x}{1 + (1 - \alpha)x} < (1 + x)^\alpha.$$

*Proof.* We rewrite this inequality as

$$1 + x + \alpha x(1 + x)^\alpha < (1 + x)^{1+\alpha}.$$

We define

$$\varphi(x) = (1 + x)^{1+\alpha} - \alpha x(1 + x)^\alpha - x - 1 \quad \text{for } x \geq 0.$$

Simple computation yields

$$\begin{aligned} \varphi'(x) &= (1 + x)^\alpha - \alpha^2 x(1 + x)^{\alpha-1} - 1, \\ \varphi''(x) &= (\alpha - \alpha^2)(1 + x)^{\alpha-1} - \alpha^2(\alpha - 1)x(1 + x)^{\alpha-2}. \end{aligned}$$

It follows that  $\varphi''(x) > 0$  for  $x > 0$  and  $0 < \alpha < 1$ ,  $\varphi'(0) = 0$  and  $\varphi(0) = 0$ . Thus,  $\varphi(x)$  is strictly increasing and  $\varphi(x) > 0$  for  $x > 0$ . This completes the proof of Lemma 2.3.  $\square$

LEMMA 2.4. *Suppose that  $r > 1$  and  $x \geq 1$ . Then*

$$(1 + x)^{\frac{1+r}{r}} - x^{\frac{1+r}{r}} > \frac{1+r}{r} x^{\frac{1}{r}}.$$

*Proof.* We rewrite this inequality as

$$(1+x)\left(1+\frac{1}{x}\right)^{\frac{1}{r}} - x > \frac{1+r}{r}.$$

This is true. Since by Lemma 2.3, we have

$$\left(1+\frac{1}{x}\right)^{\frac{1}{r}} > 1 + \frac{\frac{1}{r}x}{1+(1-\frac{1}{r})x} > 1 + \frac{1}{r(1+x)}.$$

This completes the proof of Lemma 2.4.  $\square$

LEMMA 2.5. We have

(i)  $2^{\frac{1}{r}} < \frac{1+r}{r}$  for  $r > 1$ .

(ii)  $\frac{r}{1+r}2^{\frac{1+r}{r}} > \frac{1}{r}(1+r)^{\frac{1+r}{r}}$  for  $r > \frac{26}{5}$ .

*Proof.* (i) By Bernoulli's inequality we have

$$2^{\frac{1}{r}} = (1+1)^{\frac{1}{r}} < \frac{1+r}{r}.$$

(ii) We rewrite this inequality as

$$\frac{r^2}{1+r} > \left(\frac{1+r}{2}\right)^{\frac{1+r}{r}}.$$

By Bernoulli's inequality, it follows

$$\begin{aligned} \left(\frac{1+r}{2}\right)^{\frac{1+r}{r}} &= \frac{1+r}{2} \left(1+\frac{r-1}{2}\right)^{\frac{1}{r}} \\ &< \frac{1+r}{2} \left(1+\frac{r-1}{2r}\right) = -\frac{r^2(r-5)-r+1}{4r(1+r)} + \frac{r^2}{1+r} \\ &< \frac{r^2}{1+r} \end{aligned}$$

for  $r > \frac{26}{5}$ .

This completes the proof of Lemma 2.5.  $\square$

*Proof of Theorem 2.1.* Let  $r > 1$  and set  $p = -\frac{1}{r}$ ,  $a = 1$ ,  $b = x > 1$ ,  $f(x) = a_n$ ,  $g_n(x) = (x-1)^{\frac{1}{(1+r)r}}$  for  $x \in [n, n+1]$  and  $n \in \mathbf{N}$ . Hölder's inequality then yields

$$\left(\int_1^x f(t)g(t)dt\right)^{-\frac{1}{r}} \leq \left(\int_1^x f^{-\frac{1}{r}}(t)dt\right) \left(\int_1^x g^{\frac{1}{1+r}}(t)dt\right)^{-\frac{1+r}{r}}.$$

It follows that

$$\begin{aligned} \left( \int_1^x f^{-r}(t) dt \right)^{-\frac{1}{r}} &= \left( \int_1^x ((t-1)^{1+r} f(t))^{-r} ((t-1)^{(1+r)r}) dt \right)^{-\frac{1}{r}} \\ &< \left( \int_1^x (t-1)^{1+r} f(t) dt \right) \left( \int_1^x (t-1)^r dt \right)^{-\frac{1+r}{r}} \\ &< (1+r)^{\frac{1+r}{r}} (x-1)^{-\frac{(1+r)^2}{r}} \int_1^x (t-1)^{1+r} f(t) dt. \end{aligned}$$

Then we have

$$\left( \frac{x-1}{\int_1^x f^{-r}(t) dt} \right)^{1/r} < (1+r)^{\frac{1+r}{r}} (x-1)^{-r-2} \int_1^x (t-1)^{1+r} f(t) dt.$$

Then we obtain

$$\begin{aligned} \int_1^\infty \left( \frac{x-1}{\int_1^x f^{-r}(t) dt} \right)^{1/r} dx &< (1+r)^{\frac{1+r}{r}} \int_1^\infty (x-1)^{-r-2} \int_1^x (t-1)^{1+r} f(t) dt dx \\ &= (1+r)^{\frac{1+r}{r}} \int_1^\infty \left( \int_t^\infty (x-1)^{-r-2} dx \right) (t-1)^{1+r} f(t) dt \\ &= (1+r)^{1/r} \int_1^\infty f(t) dt \\ &= (1+r)^{1/r} \sum_{n=1}^\infty a_n. \end{aligned}$$

By the definition of  $f(x)$ , Lemmas 2.3, 2.4 and 2.5, we have

$$\begin{aligned} \int_1^\infty \left( \frac{x-1}{\int_1^x f^{-r}(t) dt} \right)^{1/r} dx &> \int_1^2 \left( \frac{x-1}{\int_1^x f^{-r}(t) dt} \right)^{1/r} dx + \int_2^\infty \left( \frac{x-1}{\int_1^x f^{-r}(t) dt} \right)^{1/r} dx \\ &> \frac{\int_1^2 (x-1)^{1/r} dx}{a_1^{-1}} + \sum_{n=2}^\infty \frac{\int_n^{n+1} (x-1)^{1/r} dx}{(\sum_{k=1}^n a_k^{-r})^{1/r}} \\ &> \frac{\frac{r}{1+r}}{a_1^{-1}} + \sum_{n=2}^\infty \frac{\frac{r}{1+r} (n^{\frac{1+r}{r}} - (n-1)^{\frac{1+r}{r}})}{(\sum_{k=1}^n a_k^{-r})^{1/r}} \\ &> \frac{\frac{r}{1+r}}{a_1^{-1}} + \sum_{n=2}^\infty \frac{(n-1)^{1/r}}{(\sum_{k=1}^n a_k^{-r})^{1/r}} \\ &> \frac{\frac{r}{1+r}}{a_1^{-1}} + \left( \frac{1}{2} \right)^{1/r} \sum_{n=2}^\infty \frac{n^{1/r}}{(\sum_{k=1}^n a_k^{-r})^{1/r}} \\ &> \frac{r}{1+r} \sum_{n=1}^\infty \left( \frac{n}{\sum_{k=1}^n a_k^{-r}} \right)^{1/r}. \end{aligned}$$

This completes the proof of Theorem 2.1.  $\square$

REMARK 1. By Theorem 2.1 and Lemma 2.5 (ii), we know that the constant factor  $\frac{r}{1+r} 2^{\frac{1+r}{r}}$  in (8) is not the best possible for  $r > \frac{26}{5}$ . We give a strengthened Hardy-Carleman inequality (9) for a negative exponent.

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